CSx25: Digital Signal Processing
NCS224: Signals and Systems

Dr. Ahmed Shalaby

http://bu.edu.eg/staff/ahmedshalaby14#

Outline

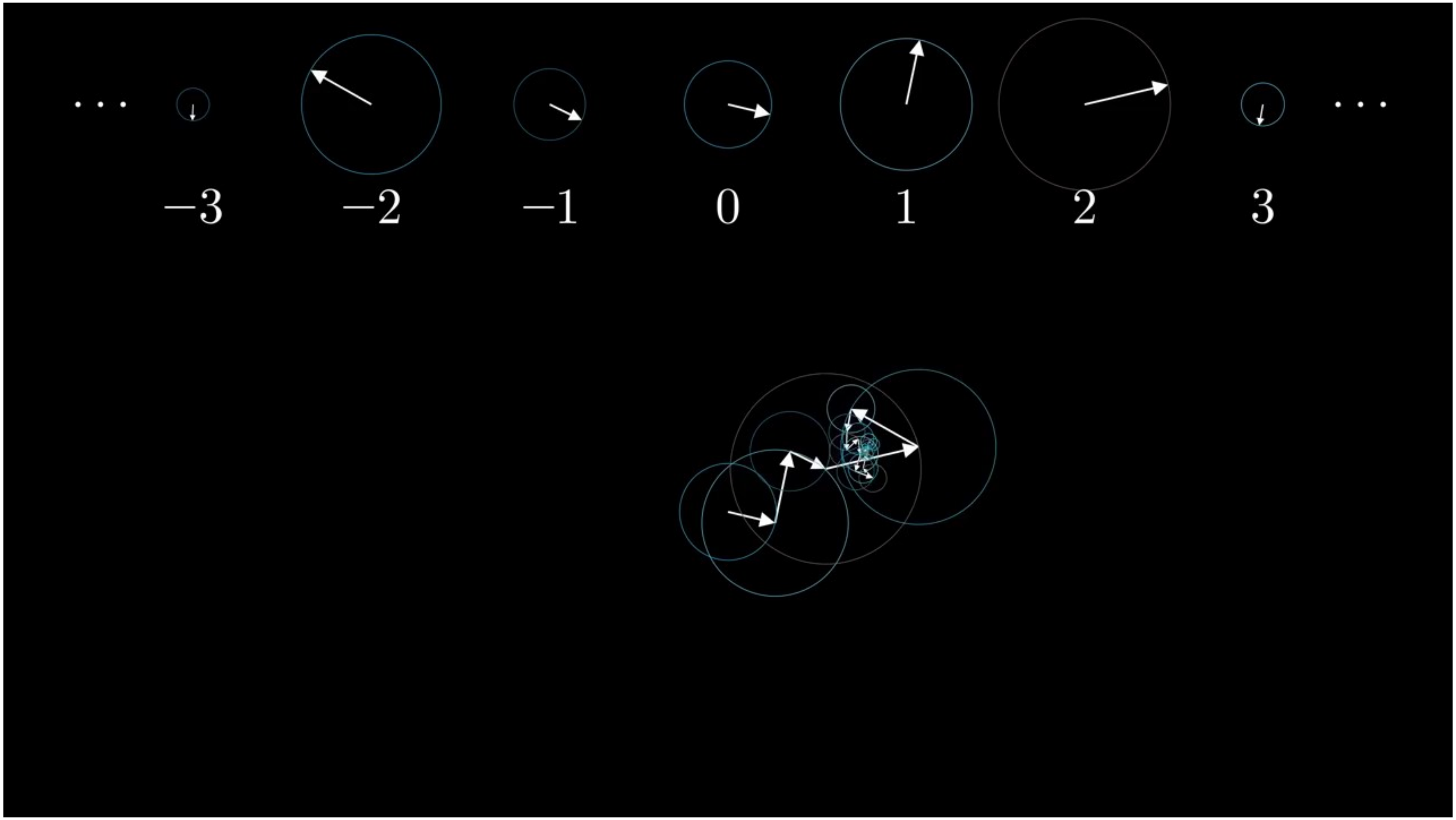
- ~~Digital Signal Processing Introduction~~
  - ~~Mathematical modeling~~
  - ~~Continuous Time Signals~~
  - ~~Discrete Time Signals~~
- ~~Analyzing Continuous-Time Systems in the Time Domain~~
- ~~Analyzing Discrete Systems in the Time Domain~~
- **Fourier Analysis for Continuous-Time Signals and Systems**

4.1 Introduction

Representing a signal as a *linear combination of single-frequency* building blocks allows us to develop a frequency-domain view of a signal that is particularly useful in understanding signal behavior and signal-system interaction problems.

If a signal can be expressed as a superposition of single-frequency components, knowing how a ***linear and time-invariant*** system responds to each individual component *helps us understand overall system behavior* in response to the signal. This is the essence of frequency-domain analysis.

4.1 Introduction



[But what is a Fourier series? From heat flow to drawing with circles | DE4 - YouTube]

4.2 Analysis of Periodic Continuous-Time Signals

Analysis of periodic continuous-time signals

Most periodic continuous-time signals encountered in engineering problems can be expressed as linear combinations of sinusoidal *basis functions*.

The basis functions can be

- individual sine and cosine functions, or
- complex exponential functions.

Techniques to be studied:

- Trigonometric Fourier series (TFS)
- Exponential Fourier series (EFS)
- Compact Fourier series (CFS)

4.2 Analysis of Periodic Continuous-Time Signals

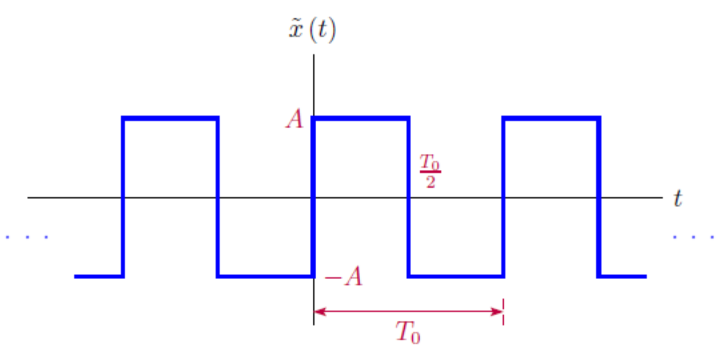
Approximating a periodic signal with trigonometric functions

Periodic signal

$$\tilde{x}(t+T_0) = \tilde{x}(t) \quad \text{for all } t$$

Additionally: A signal that is periodic with period T_0 is also periodic with kT_0 for any integer k.

An example: Periodic square wave

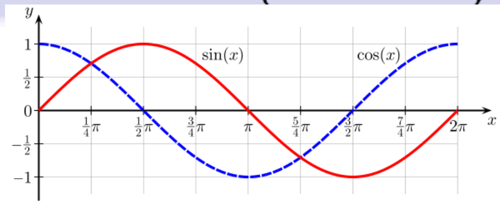


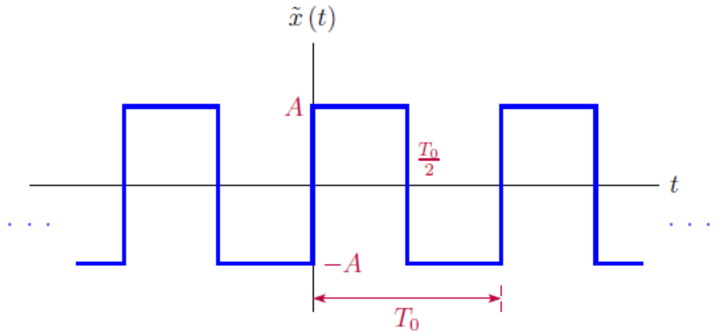
Suppose that we wish to approximate this signal using *just one trigonometric function*.

- Should we use a sine or a cosine?
- How should we adjust the parameters of the trigonometric function?

4.2 Analysis of Periodic Continuous-Time Signals

Approximating a periodic signal with trigonometric functions (continued)





The signal $\tilde{x}(t)$ has odd symmetry:

$$\tilde{x}(-t) = -\tilde{x}(t)$$

$$\left.\begin{array}{lll} \sin(-\omega t) = & -\sin(\omega t) & \text{(odd symmetry)} \\ \cos(-\omega t) = & \cos(\omega t) & \text{(even symmetry)} \end{array}\right\} \quad \Rightarrow \quad \text{Choose } \sin(\omega t)$$

$$\tilde{x}(t) \approx b_1 \sin(\omega t)$$

Signal $\tilde{x}(t)$ has a fundamental period of T_0. Pick a sine function with the same fundamental period:

$$\sin\Big(\omega(t+T_0)\Big) = \sin(\omega t) \qquad \Rightarrow \qquad \omega = \frac{2\pi}{T_0} = \omega_0 = 2\pi f_0$$

4.2 Analysis of Periodic Continuous-Time Signals

Approximating a periodic signal with trigonometric functions (continued)

$$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t)$$

$\tilde{x}^{(1)}$: "Best" approximation to $\tilde{x}(t)$ using only one frequency ω_0.

How should b_1 be chosen?

Approximation error:

$$\tilde{\varepsilon}_1(t) = \tilde{x}(t) - \tilde{x}^{(1)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t)$$

Cost function:

$$J = \int_0^{T_0} [\tilde{\varepsilon}_1(t)]^2 \, dt = \int_0^{T_0} [\tilde{x}(t) - b_1 \sin(\omega_0 t)]^2 \, dt$$

$$\text{Choose } b_1 \text{ to minimize } J \quad \Rightarrow \quad b_1 = \frac{4A}{\pi}$$

Recall that the normalized average power in a periodic signal was defined in Chapter 1 Eqn. (1.88). Adapting it to the error signal $\tilde{\varepsilon}_1(t)$ we have

$$P_\epsilon = \frac{1}{T_0}\int_0^{T_0} \left[\tilde{\varepsilon}_1(t)\right]^2\, dt \tag{4.7}$$

This is also referred to as the *mean-squared error (MSE)*. For simplicity we will drop the constant scale factor $1/T_0$ in front of the integral in Eqn. (4.7), and minimize the *cost function*

$$J = \int_0^{T_0} \left[\tilde{\varepsilon}_1(t)\right]^2\, dt = \int_0^{T_0} \left[\tilde{x}(t) - b_1\,\sin(\omega_0 t)\right]^2\, dt \tag{4.8}$$

instead. Minimizing J is equivalent to minimizing P_ϵ since the two are related by a constant scale factor. The value of the coefficient b_1 that is optimum in the sense of producing the smallest possible value for MSE is found by differentiating the cost function with respect to b_1 and setting the result equal to zero.

$$\frac{dJ}{db_1} = \frac{d}{db_1}\left[\int_0^{T_0} \left[\tilde{x}(t) - b_1\,\sin(\omega_0 t)\right]^2\, dt\right] = 0$$

Changing the order of integration and differentiation leads to

$$\frac{dJ}{db_1} = \int_0^{T_0} \frac{d}{db_1}\left[\tilde{x}(t) - b_1\,\sin(\omega_0 t)\right]^2\, dt = 0$$

Carrying out the differentiation we obtain

$$\int_0^{T_0} 2\left[\tilde{x}(t) - b_1\,\sin(\omega_0 t)\right]\left[-\sin(\omega_0 t)\right]\, dt = 0$$

or equivalently

$$-\int_0^{T_0} \tilde{x}(t)\,\sin(\omega_0 t)\, dt + b_1 \int_0^{T_0} \sin^2(\omega_0 t)\, dt = 0 \tag{4.9}$$

It can be shown that the second integral in Eqn. (4.9) yields

$$\int_0^{T_0} \sin^2(\omega_0 t)\, dt = \frac{T_0}{2}$$

Substituting this result into Eqn. (4.9) yields the optimum choice for the coefficient b_1 as

$$b_1 = \frac{2}{T_0}\int_0^{T_0} \tilde{x}(t)\,\sin(\omega_0 t)\, dt \tag{4.10}$$

For the square-wave signal $\tilde{x}(t)$ in Fig. 4.1 we have

$$b_1 = \frac{2}{T_0}\int_0^{T_0/2} (A)\,\sin(\omega_0 t)\, dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} (-A)\,\sin(\omega_0 t)\, dt = \frac{4A}{\pi} \tag{4.11}$$

4.2 Analysis of Periodic Continuous-Time Signals

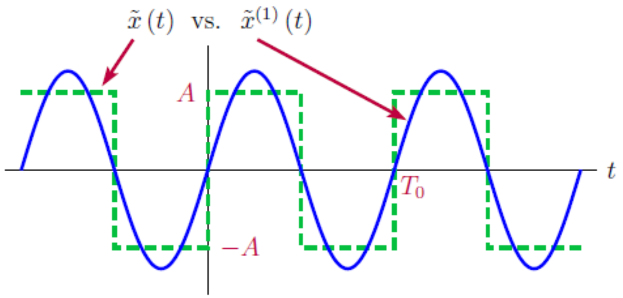
Approximating a periodic signal with trigonometric functions (continued)

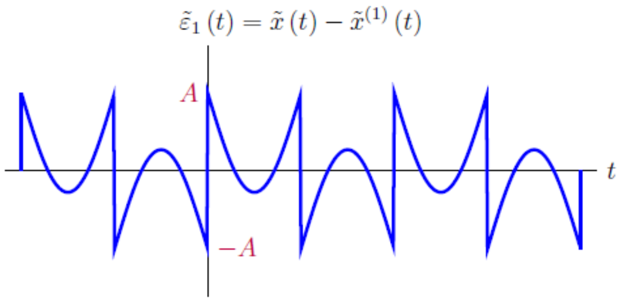
The best approximation to $\tilde{x}(t)$ using only one frequency is

$$\tilde{x}^{(1)}(t) = \frac{4A}{\pi}\sin(\omega_0 t)$$

and the approximation error is

$$\tilde{\varepsilon}_1(t) = \tilde{x}(t) - \frac{4A}{\pi}\sin(\omega_0 t)$$





4.2 Analysis of Periodic Continuous-Time Signals

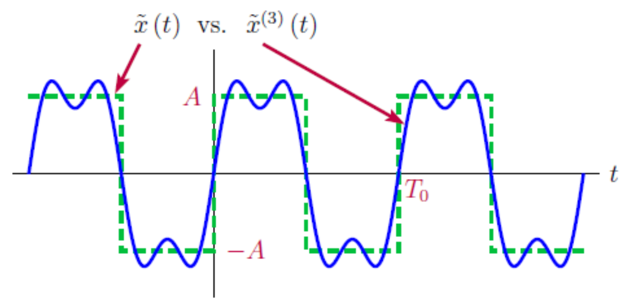
Approximating a periodic signal with trigonometric functions (continued)

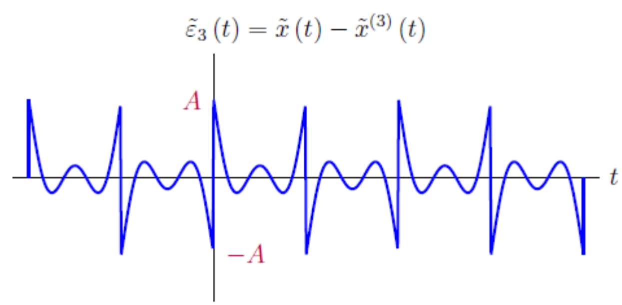
Using three frequencies:

$$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$$

Optimum coefficient values:

$$b_1 = \frac{4A}{\pi}\,, \quad b_2 = 0\,, \quad \text{and} \quad b_3 = \frac{4A}{3\pi}$$





4.2 Analysis of Periodic Continuous-Time Signals

4.2.2 Trigonometric Fourier series (TFS)

We are now ready to generalize the results obtained in the foregoing discussion about approximating a signal using trigonometric functions. Consider a signal $\tilde{x}(t)$ that is periodic with fundamental period T_0 and associated fundamental frequency $f_0 = 1/T_0$. We may want to represent this signal using a linear combination of sinusoidal functions in the form

$$\begin{aligned}\tilde{x}(t) =& a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \ldots + a_k \cos(k\omega_0 t) \ldots \\ &+ b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \ldots + b_k \sin(k\omega_0 t) + \ldots \end{aligned} \tag{4.25}$$

or, using more compact notation

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \tag{4.26}$$

where $\omega_0 = 2\pi f_0$ is the fundamental frequency in rad/s. Eqn. (4.26) is referred to as the *trigonometric Fourier series (TFS)* representation of the periodic signal $\tilde{x}(t)$, and the sinusoidal functions with radian frequencies of $\omega_0, 2\omega_0, \ldots, k\omega_0$ are referred to as the *basis functions*. Thus, the set of basis functions includes

$$\phi_k(t) = \cos(k\omega_0 t) \;, \qquad k = 0, 1, 2, \ldots, \infty \tag{4.27}$$

and

$$\psi_k(t) = \sin(k\omega_0 t) \;, \qquad k = 1, 2, \ldots, \infty \tag{4.28}$$

Using the notation established in Eqns. (4.27) and (4.28), the series representation of the signal $\tilde{x}(t)$ given by Eqn. (4.25) can be written in a more generalized fashion as

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \phi_k(t) + \sum_{k=1}^{\infty} b_k \psi_k(t) \tag{4.29}$$

4.2 Analysis of Periodic Continuous-Time Signals

Trigonometric Fourier series (TFS)

Trigonometric Fourier series (TFS)

Synthesis equation:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

Analysis equations:

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \cos(k\omega_0 t)\, dt\,, \quad \text{for} \quad k = 1, \ldots, \infty$$

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \sin(k\omega_0 t)\, dt\,, \quad \text{for} \quad k = 1, \ldots, \infty$$

$$a_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t)\, dt \qquad \text{(dc component)}$$

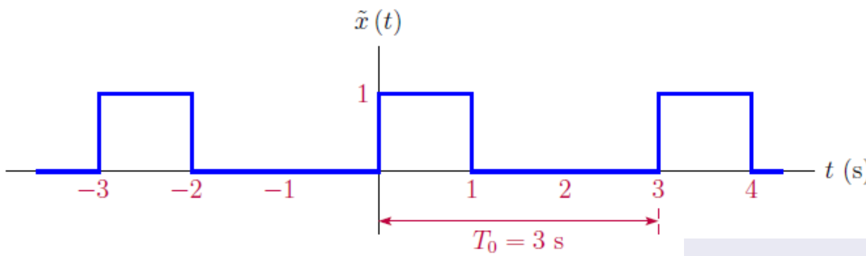
The frequencies 2ω0, 3ω0, . . . , kω0 are the second, the third, and the k-th harmonics of the fundamental frequency respectively.
The basis functions at harmonic frequencies are all periodic with a period of T0.

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.1

Trigonometric Fourier series of a periodic pulse train

A pulse train signal $\tilde{x}(t)$ with a period of $T_0 = 3$ seconds is shown. Determine the coefficients of the TFS representation of this signal.



$$a_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t)\, dt \qquad \text{(dc component)}$$

Solution:

$$a_0 = \frac{1}{3} \left[\int_0^1 (1)\, dt + \int_1^3 (0)\, dt \right] = \frac{1}{3}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{3}\ \text{Hz} \quad \Rightarrow \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{3}\ \text{rad/s}$$

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.1 (continued)

$$a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\,\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\,\sin(k\omega_0 t)\,dt$$

$$a_k = \frac{2}{3}\left[\int_0^1 (1)\,\cos(2\pi kt/3)\,dt + \int_1^3 (0)\,\cos(2\pi kt/3)\,dt\right]$$

$$= \frac{\sin(2\pi k/3)}{\pi k}\,, \qquad \text{for } k = 1, 2, \ldots, \infty$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = \boxed{-\cos x + c}$$

$$b_k = \frac{2}{3}\left[\int_0^1 (1)\,\sin(2\pi kt/3)\,dt + \int_1^3 (0)\,\sin(2\pi kt/3)\,dt\right]$$

$$= \frac{1-\cos(2\pi k/3)}{\pi k}\,, \qquad \text{for } k = 1, 2, \ldots, \infty$$

Synthesis equation:

$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\left(\frac{\sin(2\pi k/3)}{\pi k}\right)\cos(2\pi kt/3) + \sum_{k=1}^{\infty}\left(\frac{1-\cos(2\pi k/3)}{\pi k}\right)\sin(2\pi kt/3)$$

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.2

Approximation with a finite number of harmonics

Approximate the periodic pulse train of Example 4.1 using a) the first 4 harmonics, and b) the first 10 harmonics.

Solution:

Recall that

$$a_0 = \frac{1}{3}$$

$$a_k = \frac{\sin(2\pi k/3)}{\pi k}$$

$$b_k = \frac{1 - \cos(2\pi k/3)}{\pi k}$$

TFS coefficients for the pulse train:

k	a_k	b_k
0	0.3333	
1	0.2757	0.4775
2	−0.1378	0.2387
3	0.0000	0.0000
4	0.0689	0.1194
5	−0.0551	0.0955
6	0.0000	0.0000
7	0.0394	0.0682
8	−0.0345	0.0597
9	0.0000	0.0000
10	0.0276	0.0477

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.2 (continued)

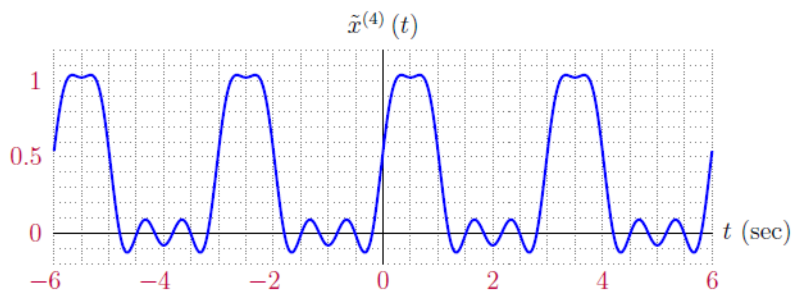
$\tilde{x}^{(m)}(t)$: Approximation to the signal $\tilde{x}(t)$ utilizing the first m harmonics.

$$\tilde{x}^{(m)}(t) = a_0 + \sum_{k=1}^{m} a_k \cos(k\omega_0 t) + \sum_{k=1}^{m} b_k \sin(k\omega_0 t)$$

Using $m = 4$:

$$\tilde{x}^{(4)}(t) = 0.3333 + 0.2757 \cos(2\pi t/3) - 0.1378 \cos(4\pi t/3) + 0.0689 \cos(8\pi t/3)$$

$$+ 0.4775 \sin(2\pi t/3) + 0.2387 \sin(4\pi t/3) + 0.1194 \sin(8\pi t/3)$$

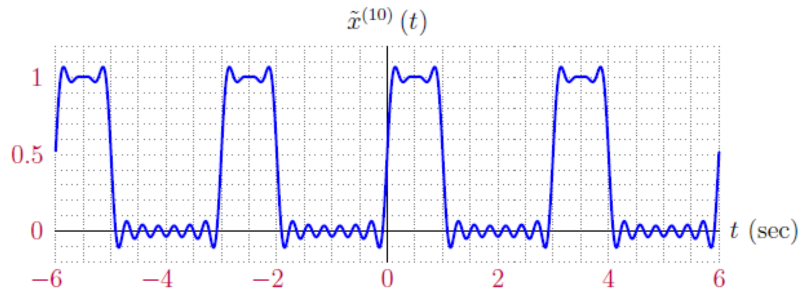


4.2 Analysis of Periodic Continuous-Time Signals

Example 4.2 (continued)

Using $m = 10$:

$$\tilde{x}^{(10)}(t) = a_0 + \sum_{k=1}^{10} a_k \cos(k\omega_0 t) + \sum_{k=1}^{10} b_k \sin(k\omega_0 t)$$



MATLAB Exercise 4.1

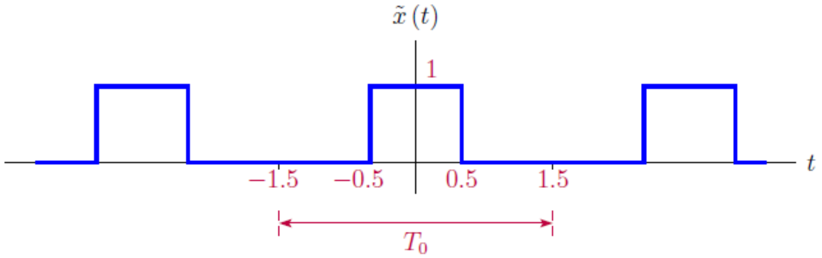
MATLAB Exercise 4.2

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.3

Periodic pulse train revisited

Determine the TFS coefficients for the periodic pulse train shown.



Solution:

- This is essentially the same pulse train used in Examples 4.1 and 4.2 except it is time shifted to center the main pulse is centered around $t = 0$.
- As a consequence, the resulting signal is an even function of time.

$$\tilde{x}(-t) = \tilde{x}(t) \quad \text{for all } t$$

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.3 (continued)

$$a_0 = \frac{1}{3}\int_{t=-0.5}^{0.5} (1)\, dt = \frac{1}{3}$$

$$a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\, dt$$

$$a_k = \frac{2}{3}\int_{-0.5}^{0.5} (1)\cos(2\pi kt/3)\, dt = \frac{2\sin(\pi k/3)}{\pi k}$$

$$b_k = \frac{2}{3}\int_{-0.5}^{0.5} (1)\sin(2\pi kt/3)\, dt = 0$$

$$a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\sin(k\omega_0 t)\, dt$$

Synthesis equation for $\tilde{x}(t)$:

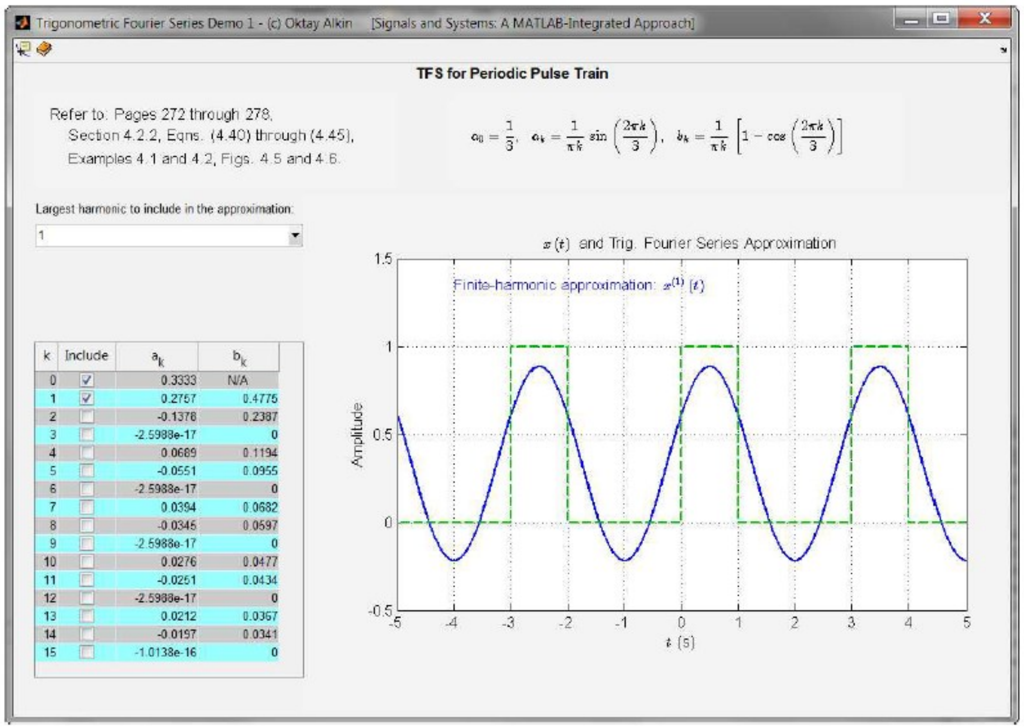
$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\left(\frac{2\sin(\pi k/3)}{\pi k}\right)\cos(k\omega_0 t)$$

Fundamental frequency: $f_0 = \frac{1}{3}$ Hz.

4.2 Analysis of Periodic Continuous-Time Signals

Interactive demo: tfs_demo1.m

Experiment with the TFS representation of periodic pulse train. Observe finite-harmonic approximations while varying the number of harmonics.



4.2 Analysis of Periodic Continuous-Time Signals

Exponential Fourier series (EFS)

Exponential Fourier series (EFS)

Synthesis equation:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t}$$

Analysis equation:

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \, e^{-jk\omega_0 t} \, dt \,, \quad k = -\infty, \ldots, \infty$$

4.2 Analysis of Periodic Continuous-Time Signals

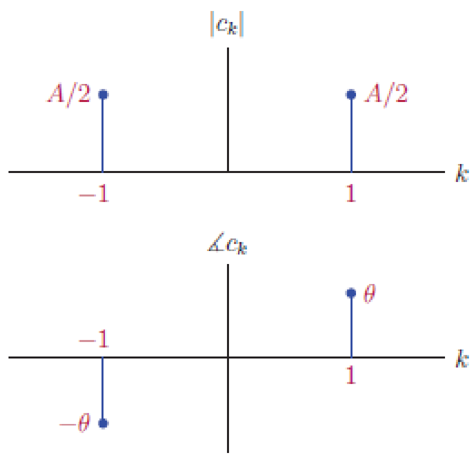
Single-tone signals:

Let us first consider the simplest of periodic signals: a single-tone signal in the form of a cosine or a sine waveform. We know that Euler's formula can be used for expressing such a signal in terms of two complex exponential functions:

$$\begin{aligned}\tilde{x}(t) =& A\cos(\omega_0 t+\theta)\\ =& \frac{A}{2}\,e^{j(\omega_0 t+\theta)}+\frac{A}{2}\,e^{-j(\omega_0 t+\theta)}\\ =& \frac{A}{2}\,e^{j\theta}\,e^{j\omega_0 t}+\frac{A}{2}\,e^{-j\theta}\,e^{-j\omega_0 t}\end{aligned} \tag{4.49}$$

Comparing Eqn. (4.49) with Eqn. (4.48) we conclude that the cosine waveform can be written in the EFS form of Eqn. (4.48) with coefficients

$$c_1=\frac{A}{2}\,e^{j\theta}\,,\quad c_{-1}=\frac{A}{2}\,e^{-j\theta}\,,\quad \text{and}\quad c_k=0\ \text{ for all other } k \tag{4.50}$$



4.2 Analysis of Periodic Continuous-Time Signals

If the signal under consideration is $\tilde{x}(t) = A\,\sin(\omega_0 t + \theta)$, a similar representation can be obtained using Euler's formula:

$$\begin{aligned}\tilde{x}(t) =& A\,\sin(\omega_0 t + \theta)\\ =& \frac{A}{2j}\,e^{j(\omega_0 t+\theta)} - \frac{A}{2j}\,e^{-j(\omega_0 t+\theta)}\end{aligned} \tag{4.51}$$

Using the substitutions

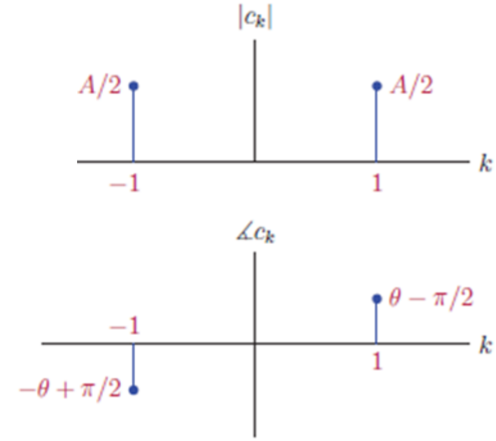
$$\frac{1}{j} = -j = e^{-j\pi/2} \quad \text{and} \quad -\frac{1}{j} = j = e^{j\pi/2}$$

Eqn. (4.51) can be written as

$$\tilde{x}(t) = \frac{A}{2}\,e^{j(\theta-\pi/2)}\,e^{j\omega_0 t} + \frac{A}{2}\,e^{-j(\theta-\pi/2)}\,e^{-j\omega_0 t} \tag{4.52}$$

Comparison of Eqn. (4.52) with Eqn. (4.48) leads us to the conclusion that the sine waveform in Eqn. (4.52) can be written in the EFS form of Eqn. (4.48) with coefficients

$$c_1 = \frac{A}{2}\,e^{j(\theta-\pi/2)}\,, \quad c_{-1} = \frac{A}{2}\,e^{-j(\theta-\pi/2)}\,, \quad \text{and} \quad c_k = 0 \;\; \text{for all other } k \tag{4.53}$$



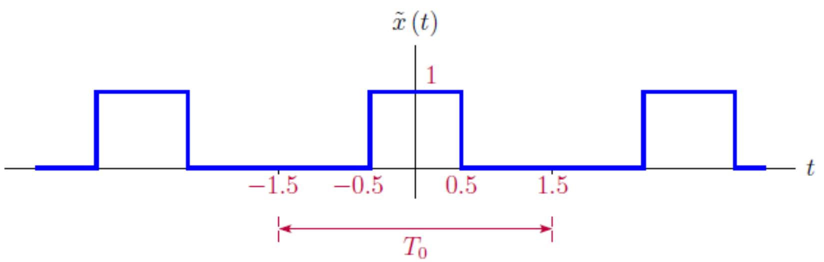
(b)

Example 4.5

Example 4.3

Exponential Fourier series for periodic pulse train

Determine the EFS coefficients of the signal $\tilde{x}(t)$ shown.



Solution:

Using $t_0 = -1.5$ and $T_0 = 3$ seconds:

$$c_k = \frac{1}{3} \int_{-0.5}^{0.5} (1)\, e^{-j2\pi k t/3}\, dt = \frac{\sin(\pi k/3)}{\pi k}$$

EFS coefficients c_k are real-valued, due to the even symmetry of $\tilde{x}(t)$.

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.5 (continued)

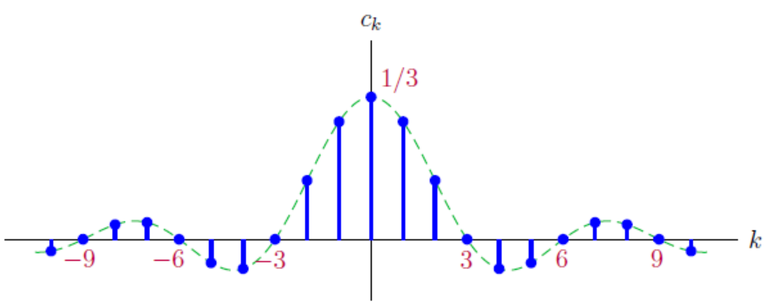
c_0 Both the numerator and the denominator = 0;

For the center coefficient use L'Hospital's rule:

$$c_0 = \left. \frac{\frac{d}{dk}\left[\sin(\pi k/3)\right]}{\frac{d}{dk}\left[\pi k\right]} \right|_{k=0} = \left. \frac{(\pi/3)\cos(\pi k/3)}{\pi} \right|_{k=0} = \frac{1}{3}$$

Synthesis equation for the signal $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left(\frac{\sin(\pi k/3)}{\pi k} \right) e^{j2\pi kt/3}$$



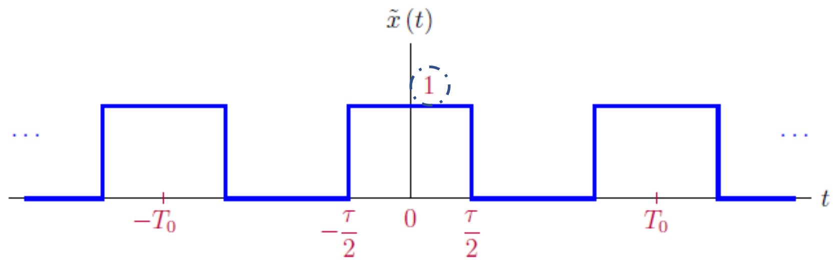
A line graph of the set of coefficients c_k is useful for illustrating the make-up of the signal $\tilde{x}(t)$ in terms of its harmonics

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.7

Effects of duty cycle on the spectrum

Determine the EFS coefficients for the pulse train shown.



Solution:

Duty-cycle is defined as: $d \triangleq \dfrac{\tau}{T_0}$

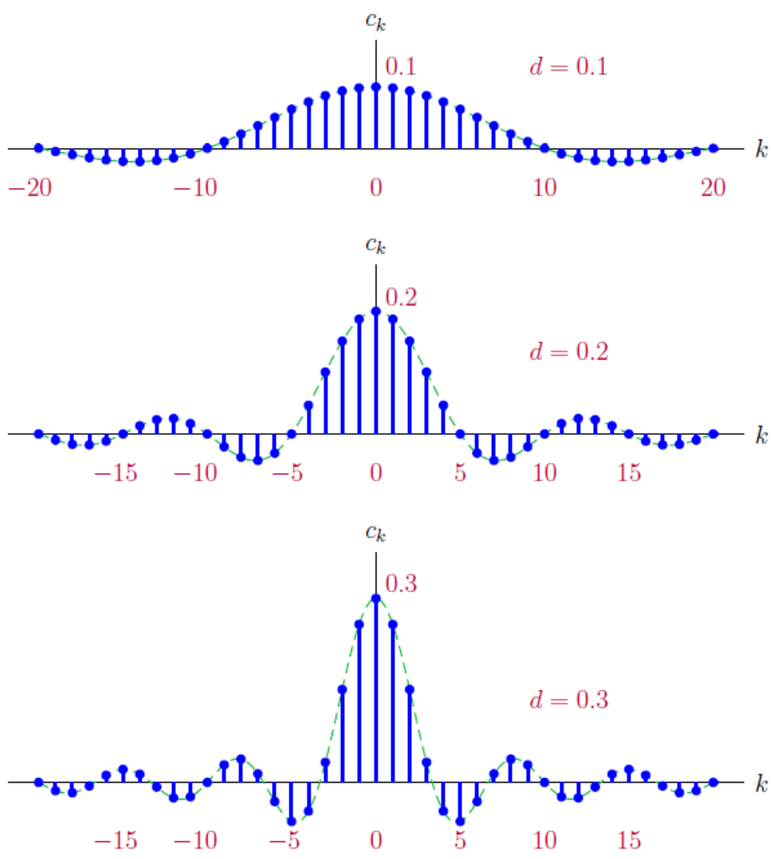
$$c_k = \frac{1}{T_0}\left[\int_{-\tau/2}^{\tau/2} (1)\, e^{-j2\pi k t/T_0}\, dt\right] = \frac{\sin(\pi k d)}{\pi k}$$

Using the sinc function:

$$c_k = d\,\mathrm{sinc}(kd)$$

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.7 (continued)



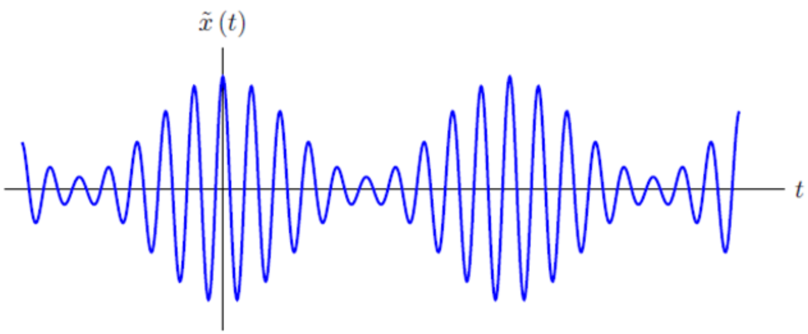
4.2 Analysis of Periodic Continuous-Time Signals

Example 4.9

Spectrum of multitone signal

Determine the EFS coefficients and graph the line spectrum for the multitone signal shown.

$$\tilde{x}(t) = \cos(2\pi [10 f_0] t) + 0.8 \cos(2\pi f_0 t) \cos(2\pi [10 f_0] t)$$



Solution:

Using the appropriate trigonometric identity $\cos(a) \cos(b) = \frac{1}{2} \cos(a+b) + \frac{1}{2} \cos(a-b)$

$$\tilde{x}(t) = \cos(2\pi [10 f_0] t) + 0.4 \cos(2\pi [11 f_0] t) + 0.4 \cos(2\pi [9 f_0] t)$$

4.2 Analysis of Periodic Continuous-Time Signals

Example 4.9 (continued)

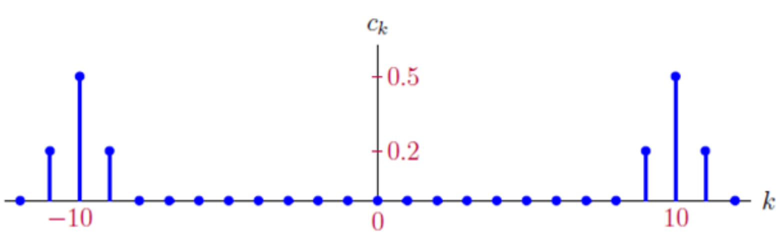
Applying Euler's formula:

$$\begin{aligned}\tilde{x}(t) =& 0.5\, e^{j2\pi(10f_0)t} + 0.5\, e^{-j2\pi(10f_0)t} \\ &+ 0.2\, e^{j2\pi(11f_0)t} + 0.2\, e^{-j2\pi(11f_0)t} + 0.2\, e^{j2\pi(9f_0)t} + 0.2\, e^{-j2\pi(9f_0)t}\end{aligned}$$

By inspection:

$$\begin{aligned} c_9 &= c_{-9} &= 0.2, \\ c_{10} &= c_{-10} &= 0.5, \\ c_{11} &= c_{-11} &= 0.2 \end{aligned}$$

All other coefficients are zero.



4.2 Analysis of Periodic Continuous-Time Signals

Compact Fourier series (CFS)

Compact Fourier series (CFS)

Synthesis equation:

$$\tilde{x}(t) = d_0 + \sum_{k=1}^{\infty} d_k \cos(k\omega_0 t + \phi_k)$$

Analysis equations:

Compute d_k and ϕ_k from TFS coefficients:

$$d_0 = a_0\ , \quad d_k = \sqrt{\left\{a_k^2 + b_k^2\right\}}\ , \quad \phi_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right) \qquad k = 1, \ldots, \infty$$

Compute d_k and ϕ_k from EFS coefficients:

$$d_0 = c_0\ , \quad d_k = 2\,|c_k|\ , \quad \phi_k = \theta_k \qquad k = 1, \ldots, \infty$$

4.2 Analysis of Periodic Continuous-Time Signals

Existence of Fourier series

1. The signal $\tilde{x}(t)$ must be integrable over one period in an absolute sense, that is

$$\int_0^{T_0} |\tilde{x}(t)|\, dt < \infty \tag{4.96}$$

   Any periodic signal in which the amplitude values are bounded will satisfy Eqn. (4.96). In addition, periodic repetitions of singularity functions such as a train of impulses repeated every T_0 seconds will satisfy it as well.
2. If the signal $\tilde{x}(t)$ has discontinuities, it must have at most a finite number of them in one period. Signals with an infinite number of discontinuities in one period cannot be expanded into Fourier series.
3. The signal $\tilde{x}(t)$ must have at most a finite number of minima and maxima in one period. Signals with an infinite number of minima and maxima in one period cannot be expanded into Fourier series.

4.2 Analysis of Periodic Continuous-Time Signals

Gibbs phenomenon

Several finite-harmonic approximations to $\tilde{x}(t)$ are shown in Fig. 4.26 for $m = 1, 3, 9, 25$. The approximation error is also shown for each case.

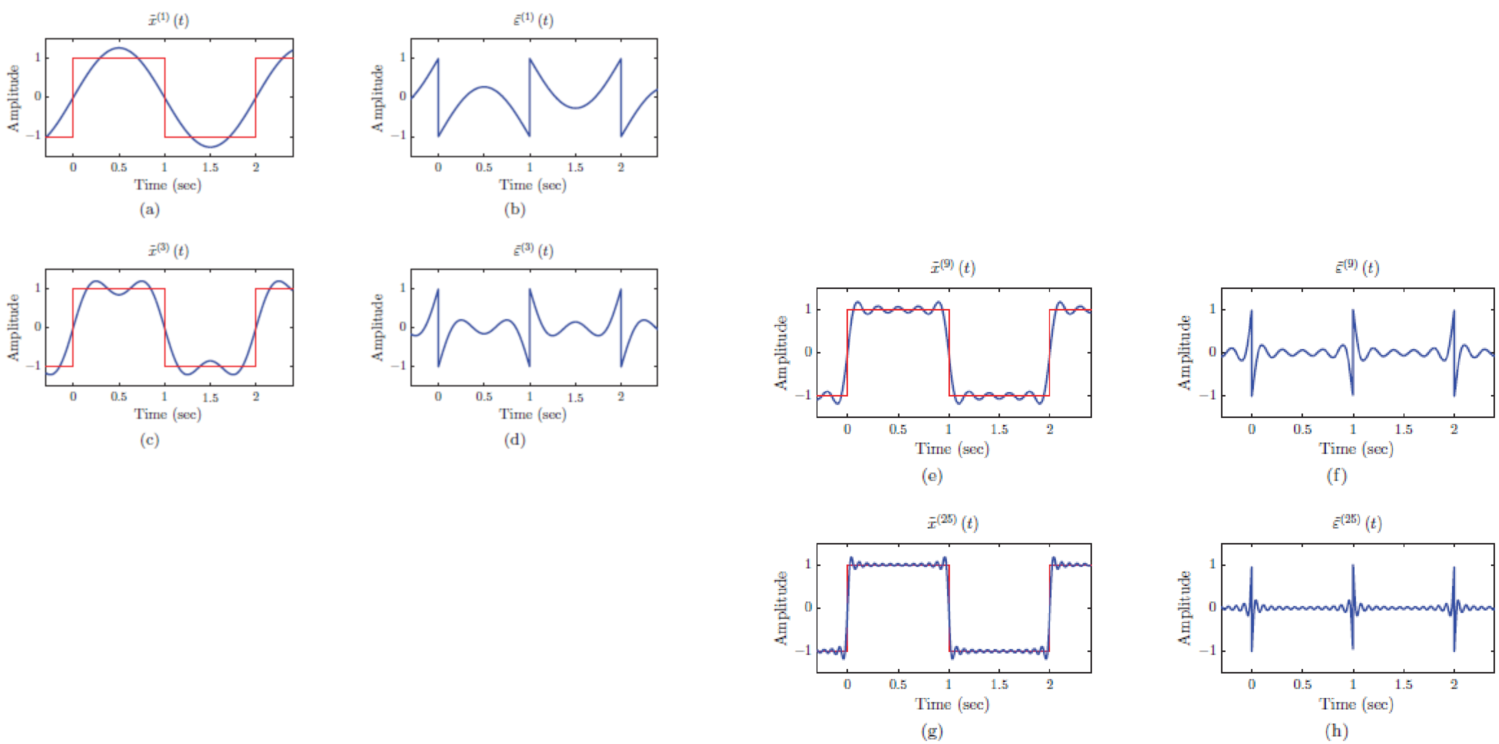


Figure 4.26 – Finite-harmonic approximations to a square-wave signal with period $T_0 = 2$ s and the corresponding approximation errors.

4.2 Analysis of Periodic Continuous-Time Signals

Properties of Fourier series

Linearity:

Consider any two signals $\tilde{x}(t)$ and $\tilde{y}(t)$ with their respective Fourier series expansions

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t} \quad \text{and} \quad \tilde{y}(t) = \sum_{k=-\infty}^{\infty} d_k \, e^{jk\omega_0 t}$$

Linearity of the Fourier series:

For any two constants α_1 and α_2

$$\alpha_1 \, \tilde{x}(t) + \alpha_2 \, \tilde{y}(t) = \sum_{k=-\infty}^{\infty} \left[\alpha_1 \, c_k + \alpha_2 \, d_k\right] e^{jk\omega_0 t}$$

4.2 Analysis of Periodic Continuous-Time Signals

Properties of Fourier series (continued)

Fourier series for even and odd signals

$$\tilde{x}(-t) = \tilde{x}(t) \text{ , all } t \quad \text{implies that} \quad \mathrm{Im}\{c_k\} = 0 \text{ , all } k$$

$$\tilde{x}(-t) = -\tilde{x}(t) \text{ , all } t \quad \text{implies that} \quad \mathrm{Re}\{c_k\} = 0 \text{ , all } k$$

Time shifting

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t} \quad \text{implies that} \quad \tilde{x}(t-\tau) = \sum_{k=-\infty}^{\infty} \left[c_k \, e^{-jk\omega_0 \tau}\right] e^{jk\omega_0 t}$$